A Model of Campanus' Sphere

In these very uncertain times, I find it comforting to look at something eternal and classical. It's even better if this is an object whose model I can construct. After months of quarantine, I finally hit upon the perfect pandemic craft object: the Campanus' Sphere.

This beautiful polyhedral frame was frequently portrayed during the Rennaissance era.

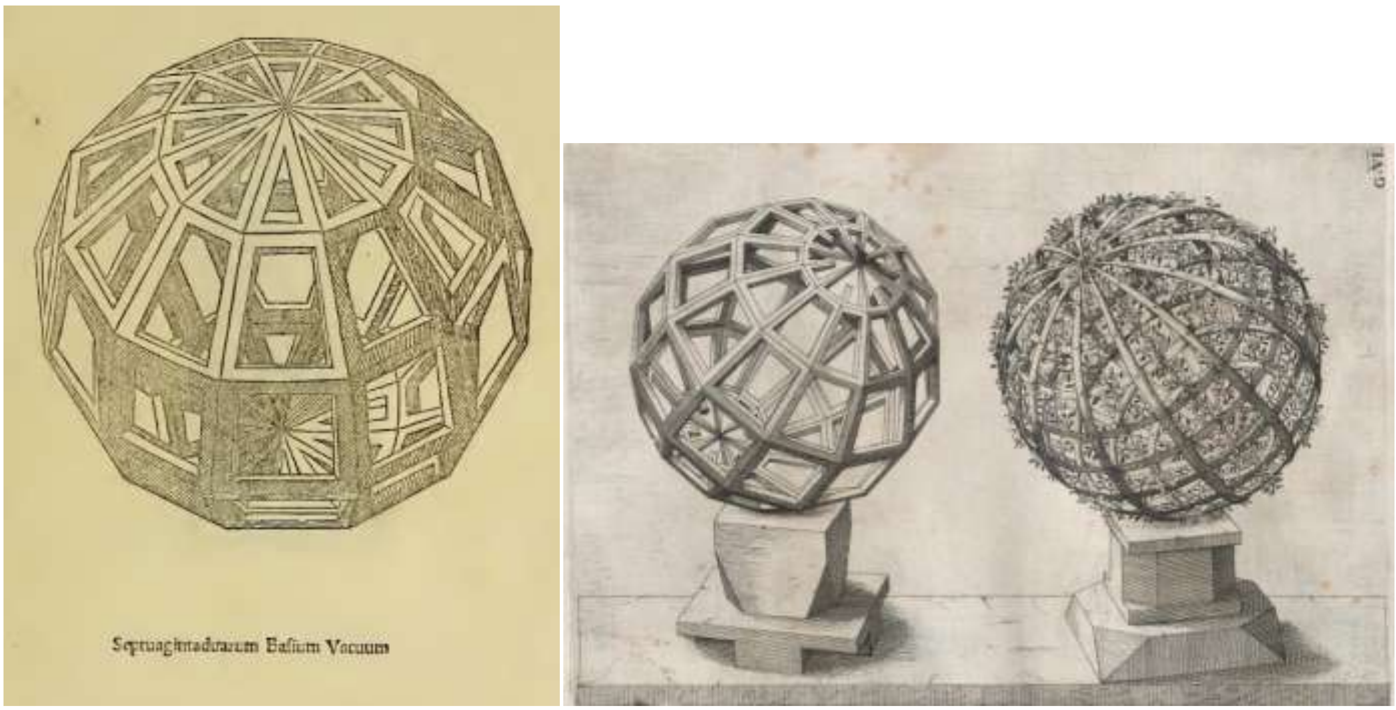

Figure 1. Renderings of the Campanus' Sphere by Leonardo da Vinci in *De Divina Proportione* (1509) and by Wenzel Jamnitzer in *Perspectiva Corpurum Regularium* (1568)

It turns out that the globe-like object is called the Campanus' Sphere, after Campanus of Novara, an Italian mathematician, astronomer, astrologer, and physician. The object was actually described by the Euclid in his *Elements (c. 300 B.C)*, but Campanus wrote a popular version of the *Elements* back in the 13thcentury, in which he described the 72-faced version of the sphere. Hence it was named after him. More details can be found in Mathematicas Visuales: http://www.matematicasvisuales.com/english/html/geometry/space/sphereCampanus.html and *Polyhedra* by Peter R. Cromwell.

The world we live in is as unpredictable as ever, and I find the idea that it can be represented by a polyhedron whose dimensions are familiar quantities from high school geometry strangely calming. The time consuming process of making the object is actually an advantage during long days of isolating at home and listening to disturbing news stories.

If you wish only to know the dimensions necessary to construct one of the models in Figure 1 and you don't care about the derivation, you can simply skip to the boxed last line of this document.

Creating a Campanus' Sphere goes like this. Start with a sphere, decide on a natural number n. The sphere will be divided into $4n$ longitude sections(each one of which will intersect with both poles), and $2n$ latitude sections. In the versions of the Campanus' Sphere shown in Figure 1, $n = 3$. Then the polygon which models the equator of the sphere will be a regular $4n$-gon inscribed in the sphere at its equatorial plane. There will be $4n$ longitude $4n$-gons going through the vertices of the equatorial polygon and both poles. All of the longitude polygons are congruent to the equatorial polygon. Finally, there will be $2n - 2$ latitude polygons parallel to the equatorial polygon but getting smaller in size as they get closer to the poles. Each of these will pass through the vertices of the longitude polygons.

When constructing a model of the Campanus' sphere, one chooses the size of the equatorial polygon randomly. The sizes of the non-equatorial latitude polygons will then need to be computed. For the $n = 3$ model, one only needs to compute 2 lengths. These are the edge lengths of the green and blue dodecagons shown in Figure 2 below.

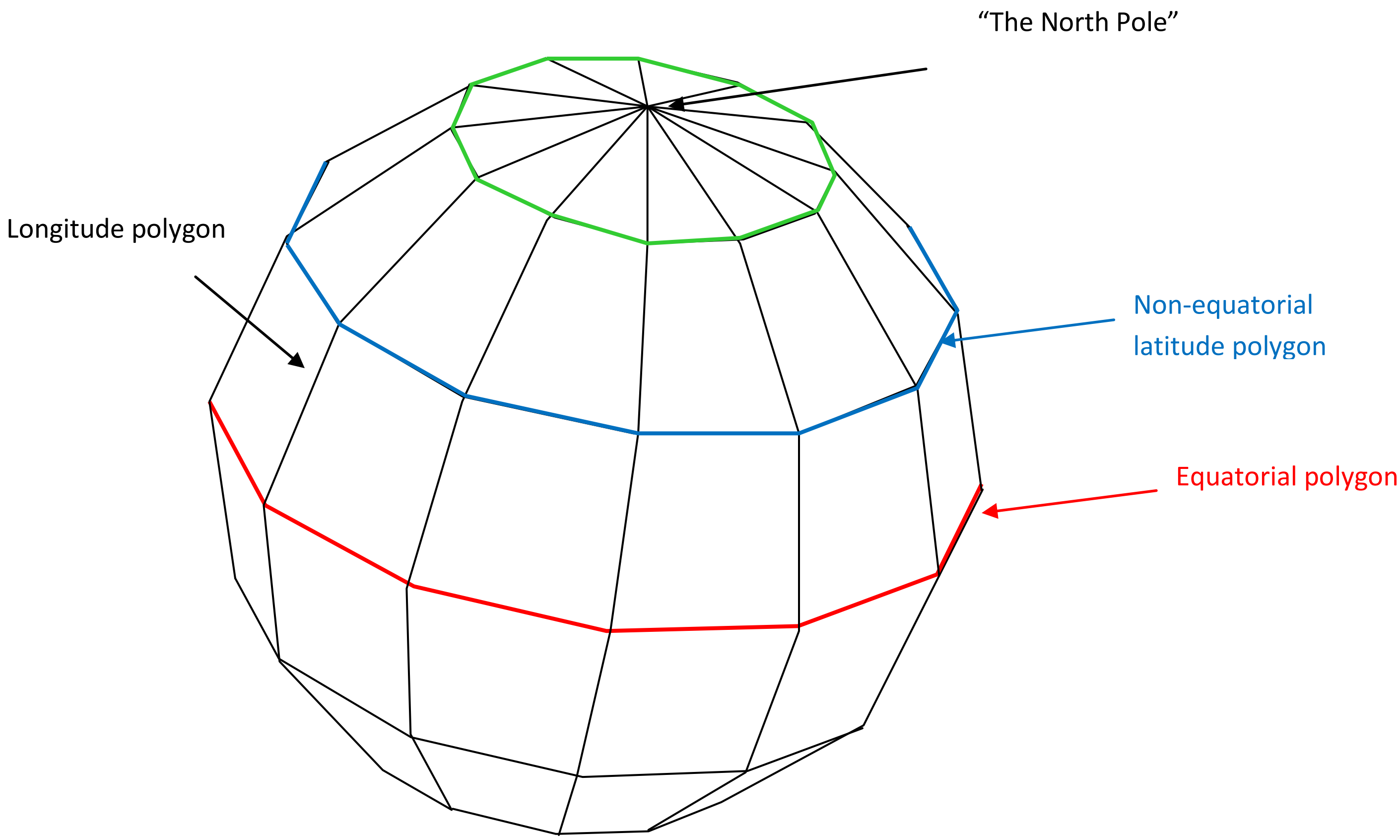

Figure 2: Campanus' Sphere for case $n = 3$

Put in a different way, we need to compute the lengths of the bases of the green and blue dotted isosceles triangles in Figure 3, denoted by s_1 and s_2 .

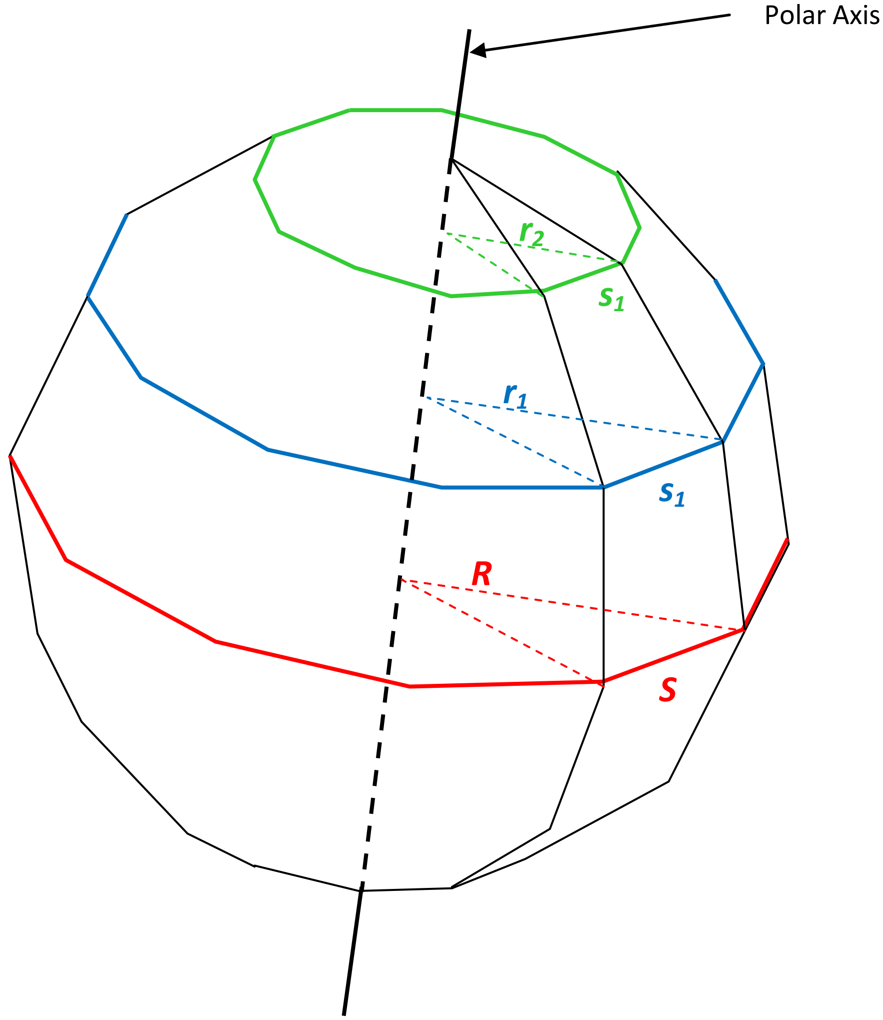

Figure 3

R is the radius of the sphere that circumscribes our Campanus' sphere. r_1 and r_2 are the radii of the latitude circles on this circumscribing sphere. To see how the lengths of ***R***, r_1 and r_2 are related, let's have a look at the factor by which the latitude circles get scaled as they get closer to one of the poles.

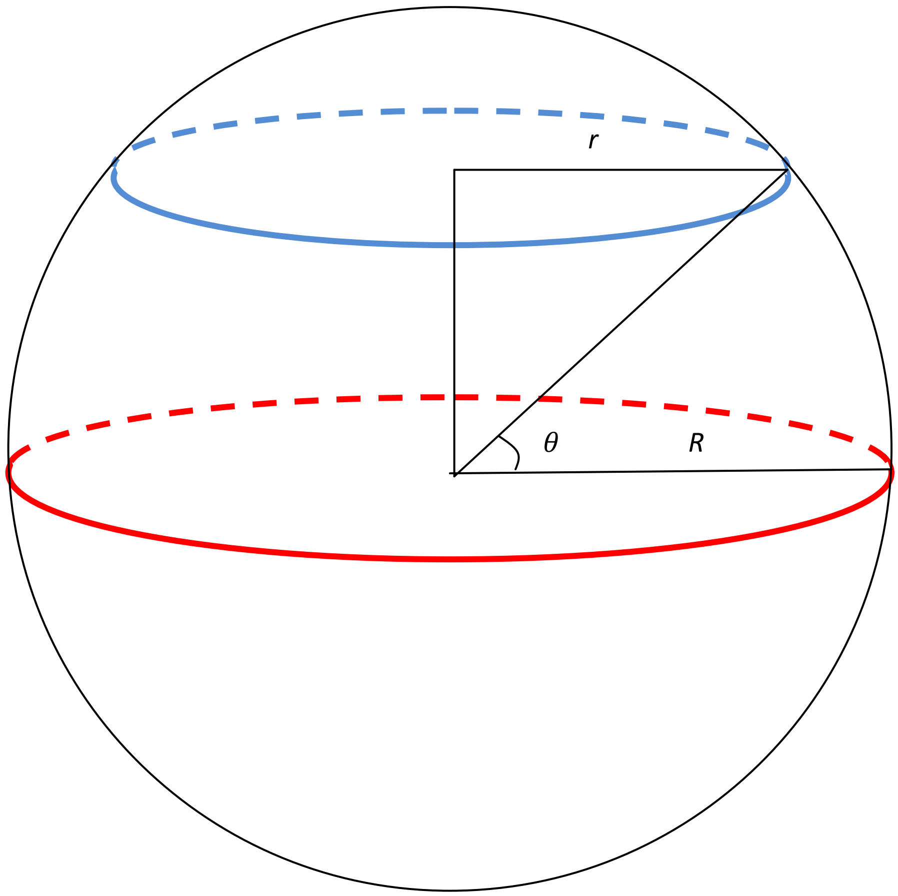

Figure 4

Consider a latitude circle with radius r shown in Figure 4. The line joining it to the center of the sphere makes an angle θ with the polar axis. This angle is what geographers call *degrees latitude*. By definition of the cosine function, $r = R\cos\theta$. This implies that $r_1 = R\cos\theta_1$, and $r_2 = R\cos\theta_2$, where $\theta_1$and $\theta_2$are the respective degrees latitude. These angles can be seen in Figure 5, where the dodecagon is one of the longitude polygons, and the colored dots represent intersections of the latitude polygons with the longitude polygon.

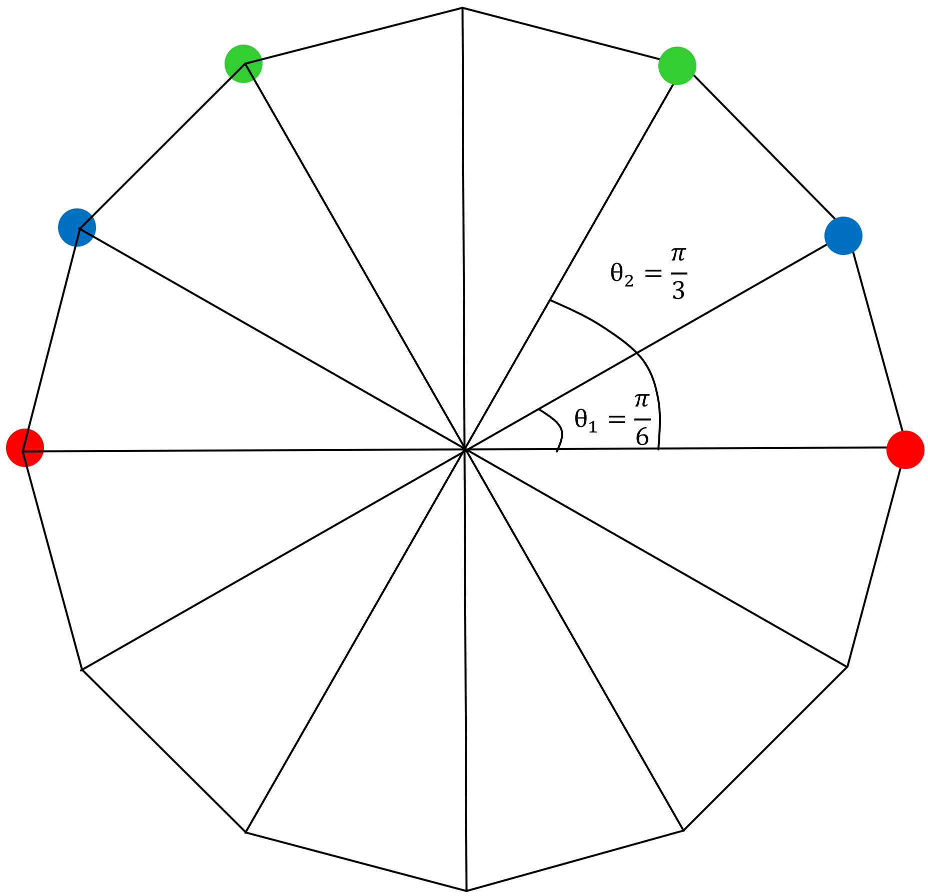

Figure 5: cross section of Figure 1 through one of the longitude polygons

The dots coincide with the vertices of a regular dodecagon. Therefore, the angle between any two adjacent radial lines is $^{\pi}/_{6}$. We therefore have that $r_1 = R\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}R$, and $r_2 = R\cos\frac{\pi}{3} = \frac{1}{2}R$.

For constructing a model of the Campanus' sphere, we are interested in the lengths of $\boldsymbol{s_1}$ and $\boldsymbol{s_2}$ in Figure 3. Since the red, blue, and green isosceles triangles all share the same apex angle, they are all similar to each other.

Therefore, $s_1 = \frac{\sqrt{3}}{2}S$ and $s_2 = \frac{1}{2}S$, where S is the arbitrarily chosen length of one of the equatorial edges in your model.