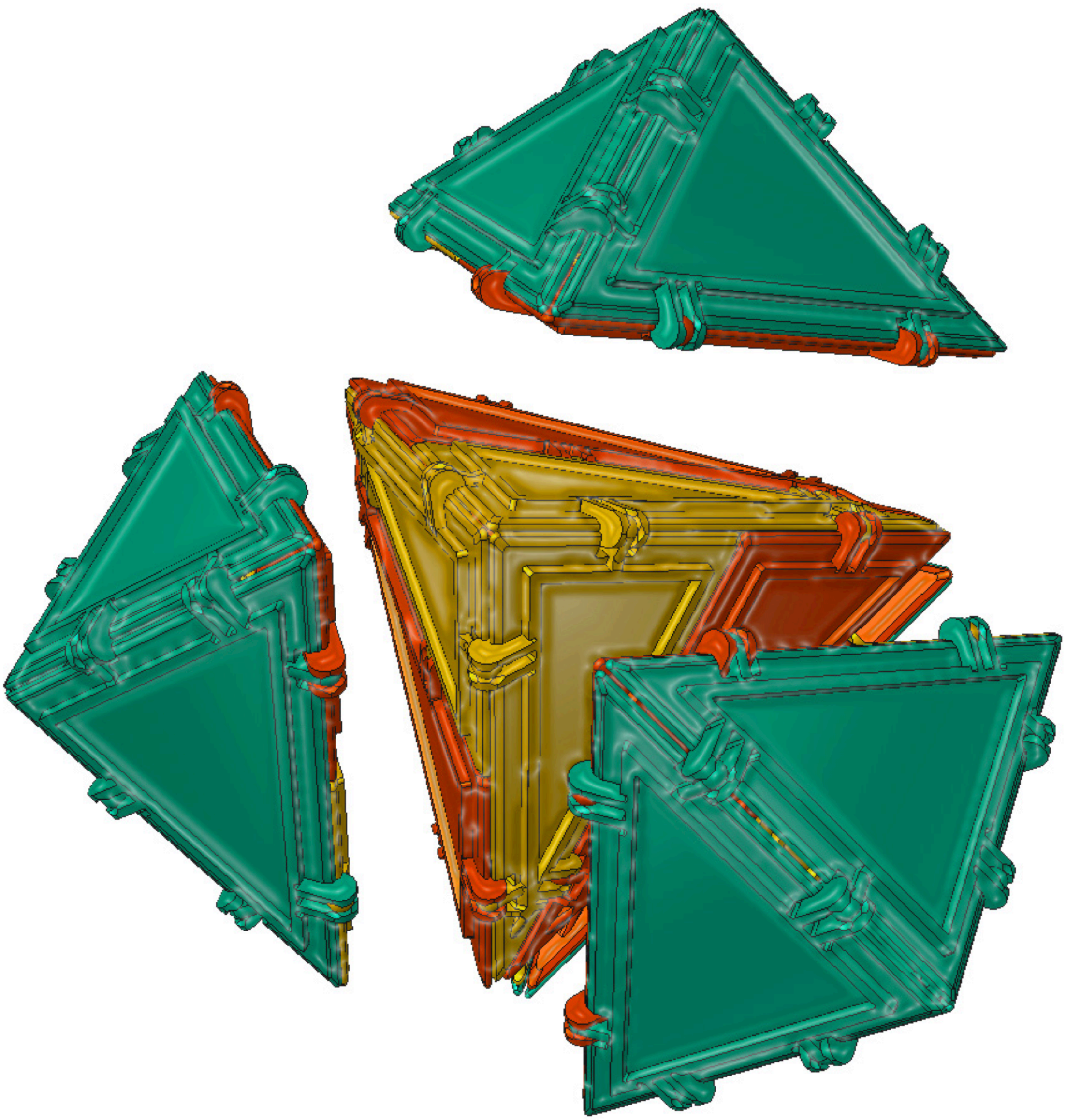
VOLUME AND CAVALIERI'S PRINCIPLE

GEOMETILES®

Volume and Cavalieri's principle

For grades 8 and up



Patent Pending

www.geometiles.com

Introduction

Cavalieri's principle is at the heart of the study of volumes, and it forms a natural transition to the use of calclulus to compute volumes. This workbook is intended to help students in grades 8 and higher develop get an informal sense of the limit-type arguments used in calculus and develop intuition about volume and surface area.

Activities in this workbook and the Common Core State Standards (CCSS)

CCSS for Mathematical Content supported by activities in this workbook

HSG.GMD.A1
Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments*.

HSG.GMD.A2
Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

HSG.GMD.A3
Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.*

CCSS for Mathematical Practice supported by activities in this workbook

MP.1 Make sense of problems and persevere in solving them.

Each problem consists of a goal (prove that two solids have the same/different volume or surface area) and a set of constraints (the two dimensional polygons from which the solid is made). Students are encouraged to explore the different ways to decompose one object and rearrange them to make another.

MP.2 Reason abstractly and quantitatively.

Students are asked to abstract the ideas behind the tangible examples of Cavalieri's principle to derive volume formulas for cones and pyramids.

MP.3 Construct viable arguments and critique the reasoning of others.

These problems lend themselves to students working together. The collaboration inevitably leads to students constructing arguments and evaluating each other's reasoning.

MP.4 Model with mathematics

Students use concrete models of rectangular prisms, parallelepipeds and tetrahedra to derive volume formulas.

MP.6 Attend to precision

Students must be accurate about comparing lengths, areas, and volumes in order to solve problems in this workbook.

Learning objectives of this workbook

The goals of this workbook are for students to:

1) Experience the concepts of volumes and surface area kinesthetically, thereby helping develop intuition and confidence in dealing with these concepts in future problems.
2) To understand Cavalieri's principle and how it is used to compute volumes
3) To understand the derivation of the volume formula for pyramids and cones.

Cavalieri's principle

Cavalieri's principle is key to deriving many volume formulas. It states that two solids have the same volume if their cross sections at the same height have the same area. The figure below shows an example of the application of Cavalieri's principle to two tetrahedra (triangular pyramids).

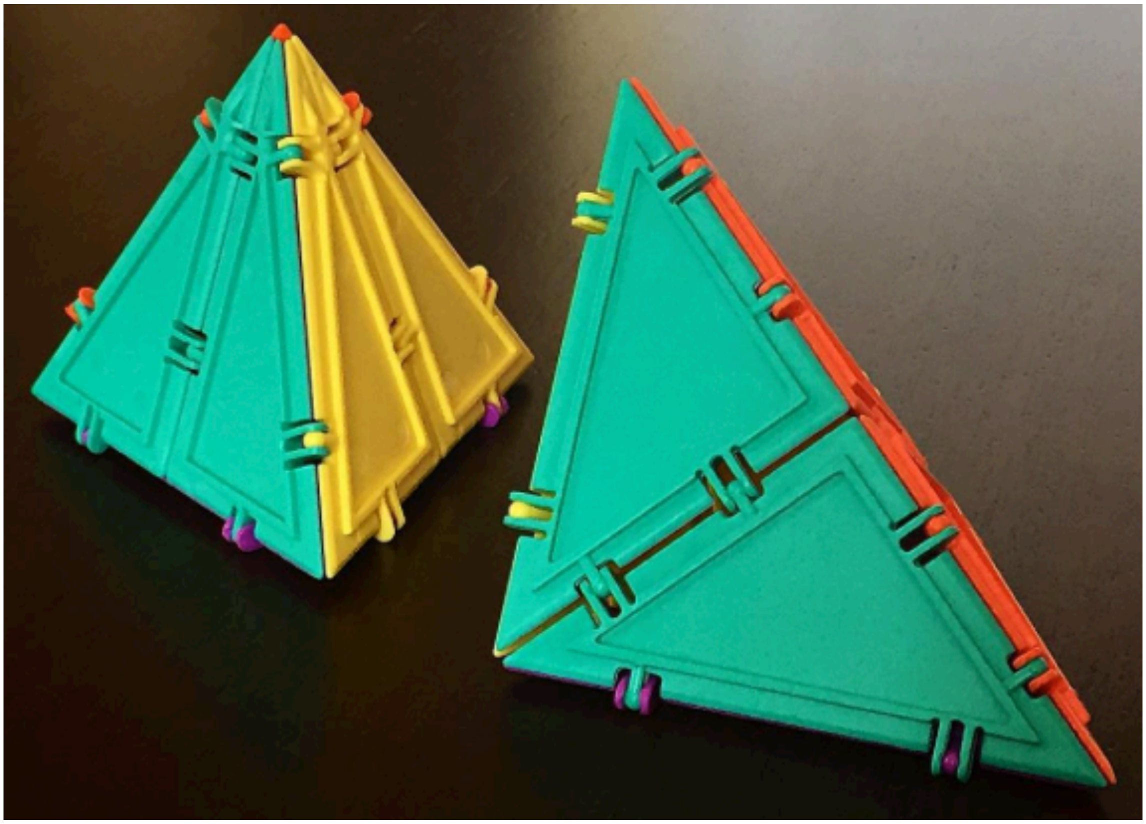


Figure 1

The tetrahedra are clearly different, yet they have the same volume because at each height, their cross sections are the same. Here is a rough idea of how this works. Let's break up each tetrahedron into thin layers of equal thickness.

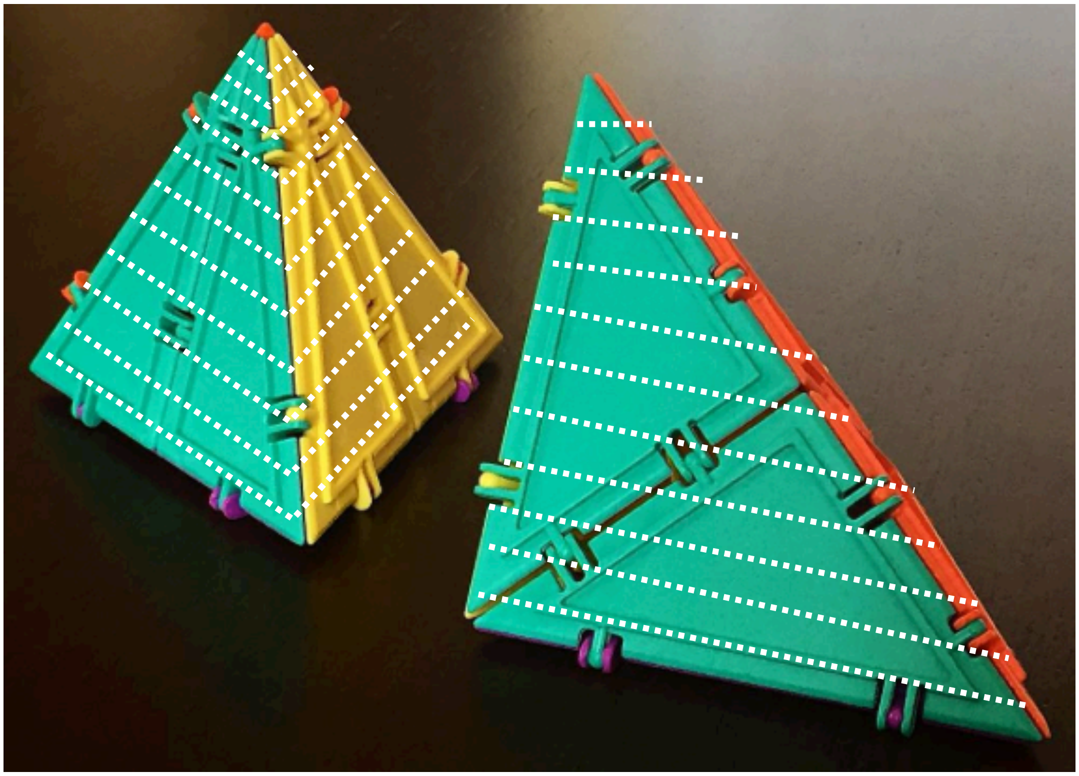


Figure 2

Now let's approximate each layer with a very short triangular prism:

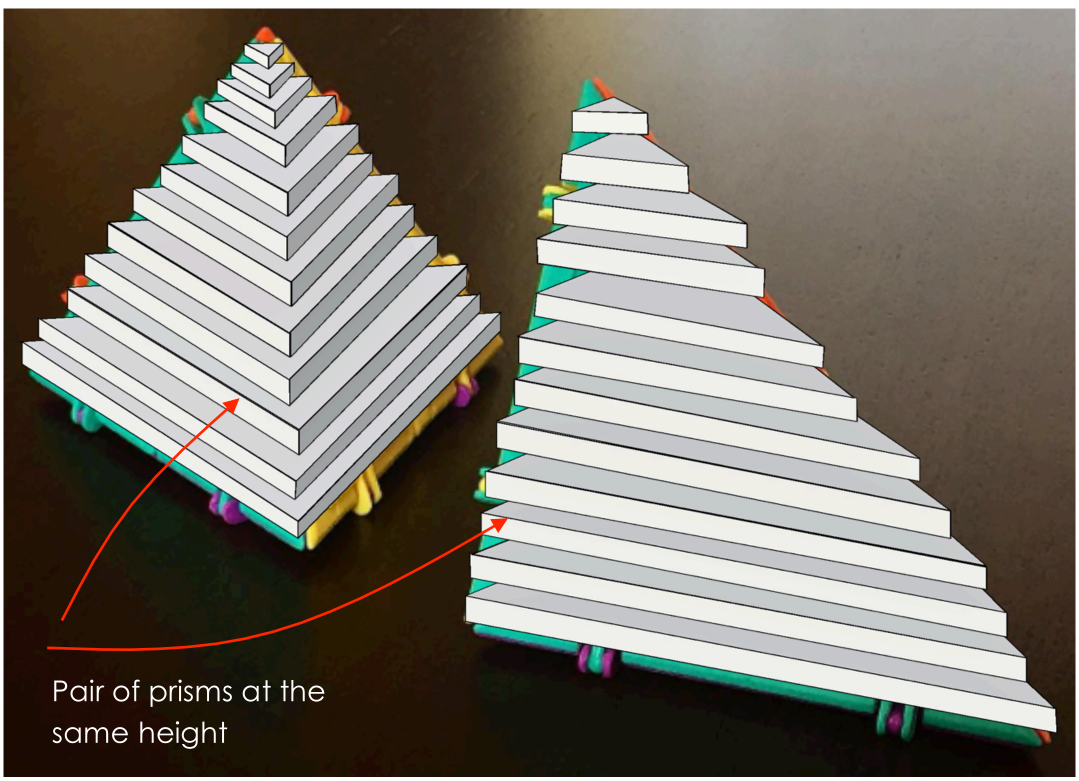
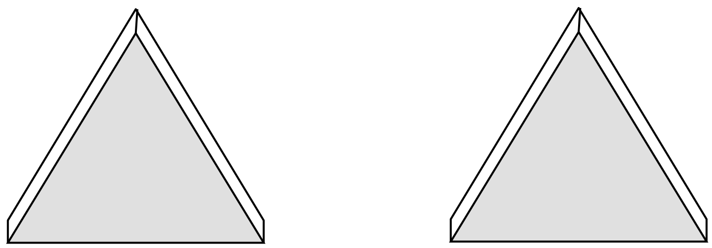


Figure 3

Now let’s take a pair of prisms, one from each tetrahedron, such that they are at the same height of each tetrahedron, and set them on their narrow side.



We find that their cross sections of are congruent equilateral triangles (due to perspective, this is not obvious from the picture). Therefore, the prisms have equal volume. They are just stacked differently in each of the two tetrahedra. But this doesn’t affect the volume of the tetrahedra, which is approximately the sum of the volumes of the short prisms.

Note: in general, the cross sections need not be congruent to each other for the application of Cavalieri’s principle; they need only have the same area.

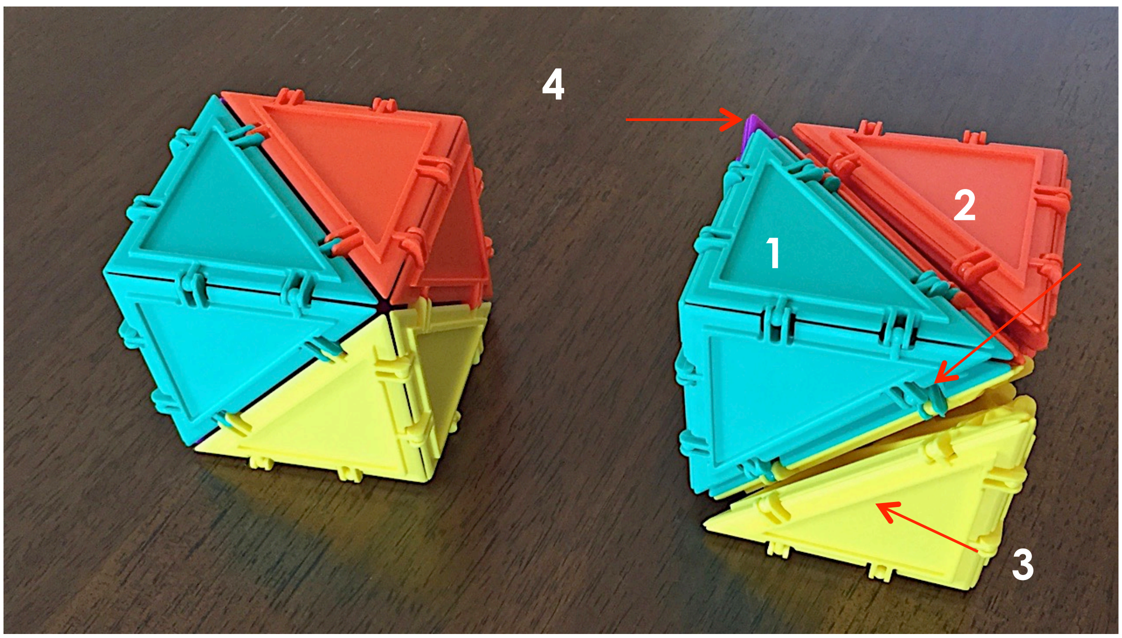
Invite your students to make each of these tetrahedra and convince themselves that they have the same volume.

Volume of all things cone-like.

Amazingly, the simple proof in the previous section is a key ingredient in deriving the famous formula for the volume of a cone or a pyramid with a polygonal base:

$$V = \frac{1}{3}(area\ of\ the\ base) \cdot height$$

We start with the picture on the cover of this workbook, which shows how a cube can be decomposed into 4 congruent tetrahedra and one regular tetrahedron. The fact that the tetrahedra make a cube may not be obvious from the picture on the cover. Here is a picture of the 5 tetrahedra taped together into a cube (right), and the same cube constructed out of isosceles triangles (left):



Regular tetrahedron (5) is inside

Figure 4

If we take the 5 tetrahedra apart and lay them out, they will look like this:

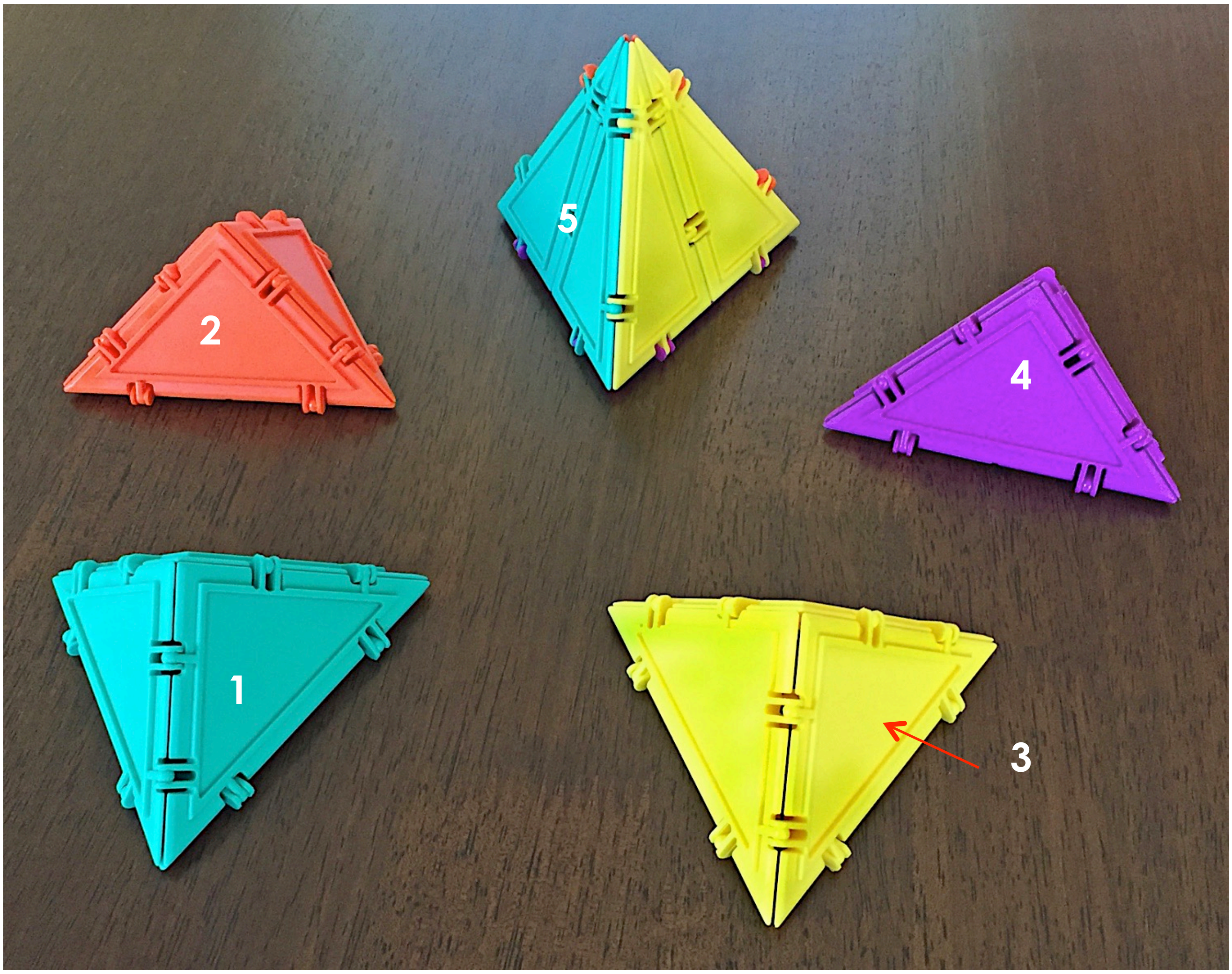


Figure 5

The numbers used to label the tetrahedra are the same as those used in Figure 4. Now you can see the regular tetrahedron inside. This is the same tetrahedron as you saw on the left of Figure 1.

Here is what happens when you attach tetrahedra 3 and 4 together.

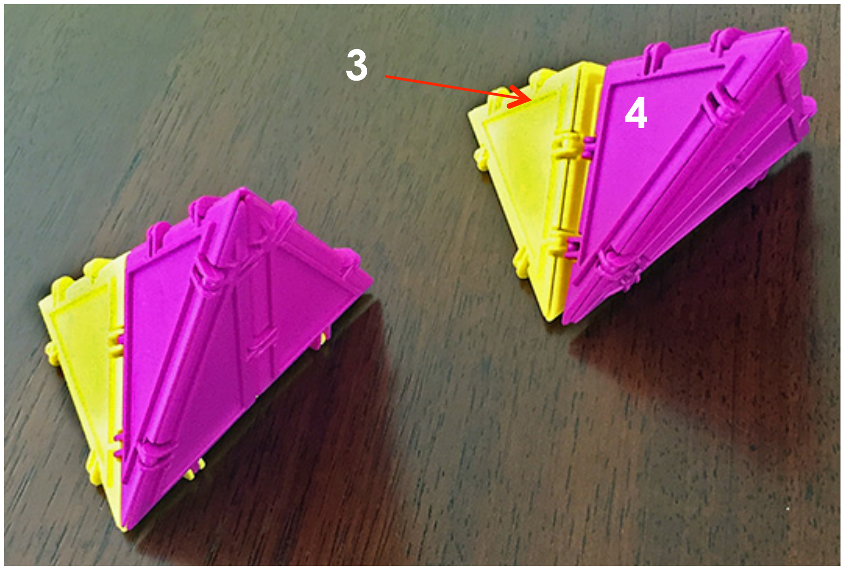


Figure 6

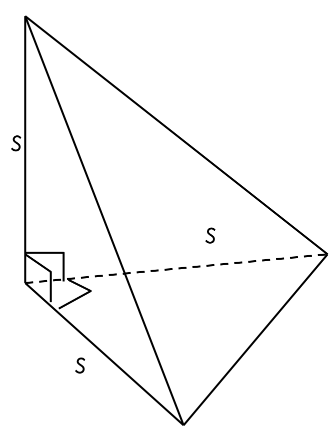
On the right we see tetrahedra 3 and 4 taped together, and on the left we see thm actually attached. The left tetrahedron is congruent to the tetrahedron on the right of Figure 1.

But we proved in the previous section that this tetrahedron has the same volume as the regular tetrahedron. Therefore, in Figure 5 the tetrahedron 5 has volume twice that of each of the tetrahedra 1-4. It follows **that the volume of one of the tetrahedra 1-4 Is 1/6 the volume of a cube.**

The volume of the cube of side length *s* is, of course, s^3. So we have just proved that the volume of one of the tetrahedra 1-4 is $\frac{1}{6}s^3$. We can rewrite this last expression as

$$\frac{1}{6}s^3 = \frac{1}{3}\left(\frac{1}{2}s^2\right)s\,.$$

Notice that $\frac{1}{2}s^2$ is the area of the base of the tetrahedron, while *s* is its height.



So we have proved that for a certain special case, the volume of a tetrahedron is

$V = \frac{1}{3}(area\ of\ the\ base) \cdot height.$

What about a more general tetrahedron? Let's start with Figure 4, and elongate it by a factor of w.

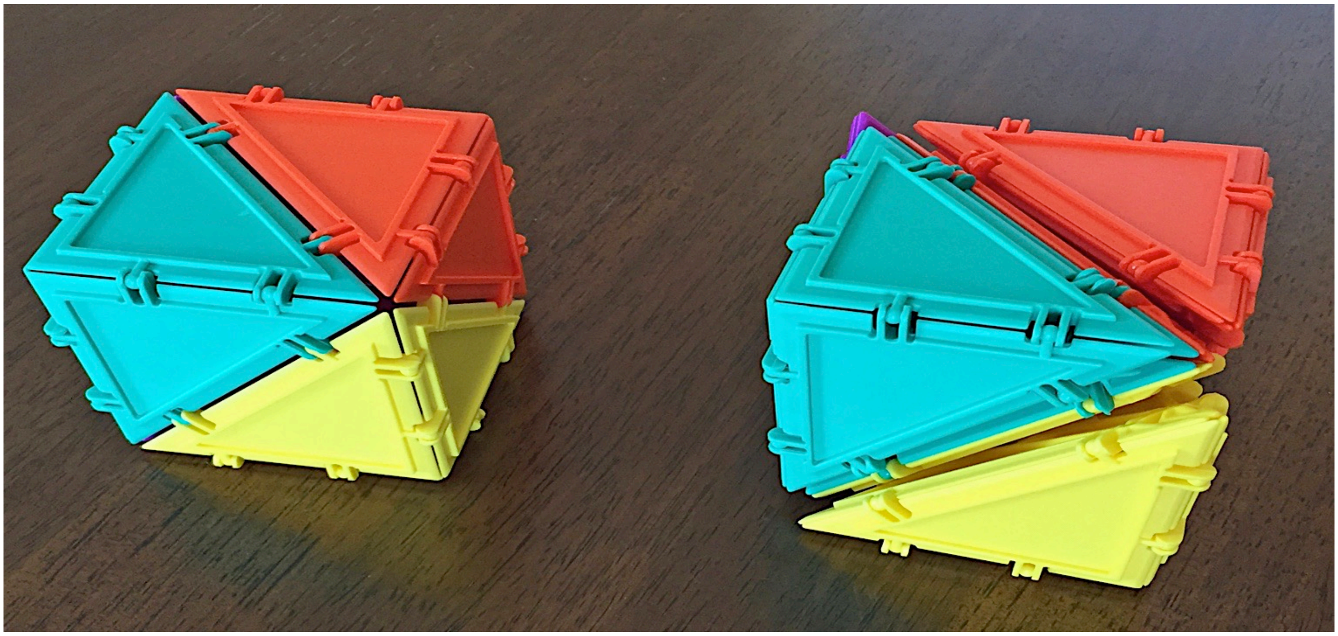
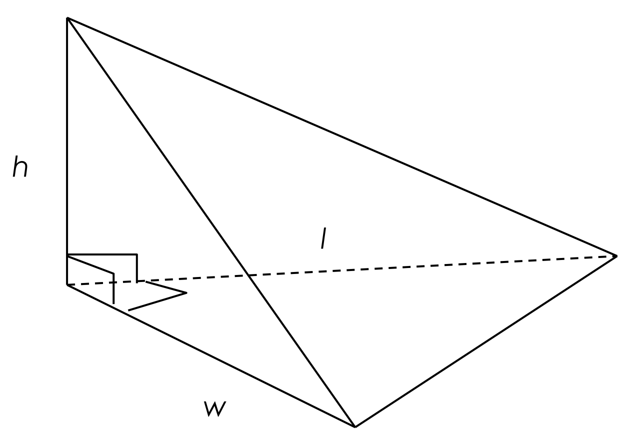


Figure 7

Every one of the solids involved (5 tetrahedra and cube) just had their volume scaled by a factor of w. So, assuming we started out with a unit cube, it is still split up into 5 tetrahedra, with the middle one having volume that's twice the volume of each of the smaller tetrahedra. We can repeat the same argument by scaling in the vertical direction by h and the other direction by l. So we actually have the more general case that the volume of this tetrahedron is also $V = \frac{1}{3}(area\ of\ the\ base) \cdot height.$



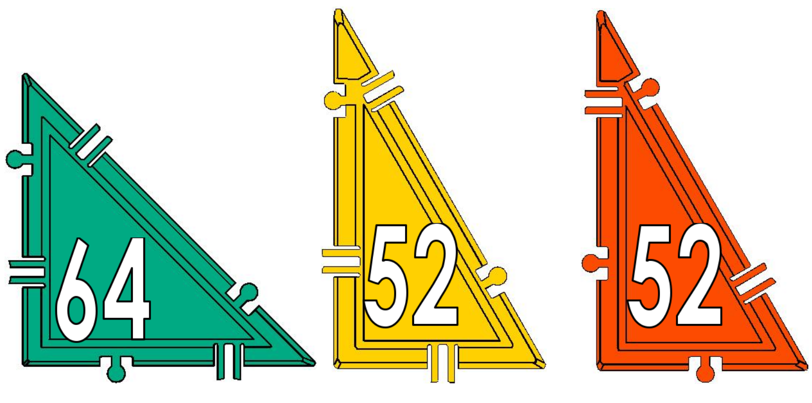
.

Since the volume of the box is *lwh,* the volume of each tetrahedron is 1/6 *lwh*. We can recognize in this formula that ½ *lw* is the area of the base of the tetrahedron, while h is its height. So we have that the area of a tetrahedron with two right angles is 1/3 area of base times height. Cavalieri's principle allows us to apply this formula to any cone-like object of the same height as our right angled tetrahedron and base of the same area (by that we mean one that comes to a point in a linear fashion, so that its cross sections at any height are the same as the cross sections of the tetrahedron).

Recommended classroom plan

The exercises are designed to work with a Jumbo Set of 512 tiles for a class with 7 groups of students working simultaneously.

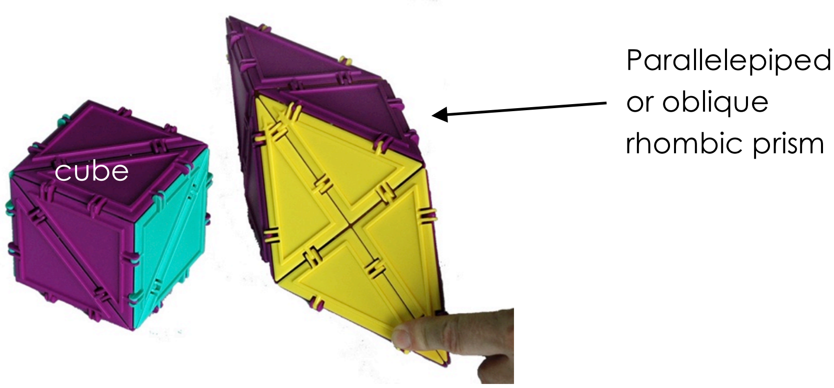
These are the shapes and quantities needed. Colors may vary.



Exercises

Exercise 1

Prove that the cube made of 12 isosceles triangles has the same volume as the parallelepiped (oblique rhombic prism)in 2 ways.

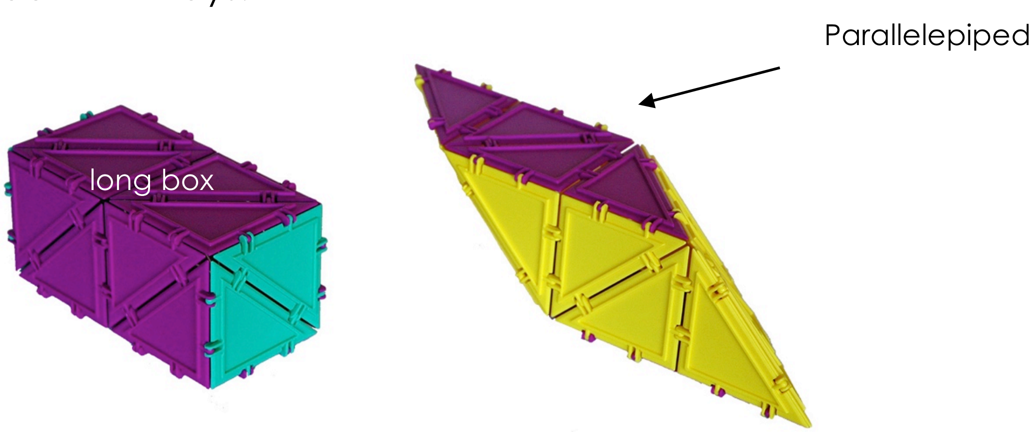


a) Cavalieri's principle
b) Dissection

Which object has the greater surface area?

Exercise 2

Prove that the parallelepiped below has the same volume as the long box in 2 ways.

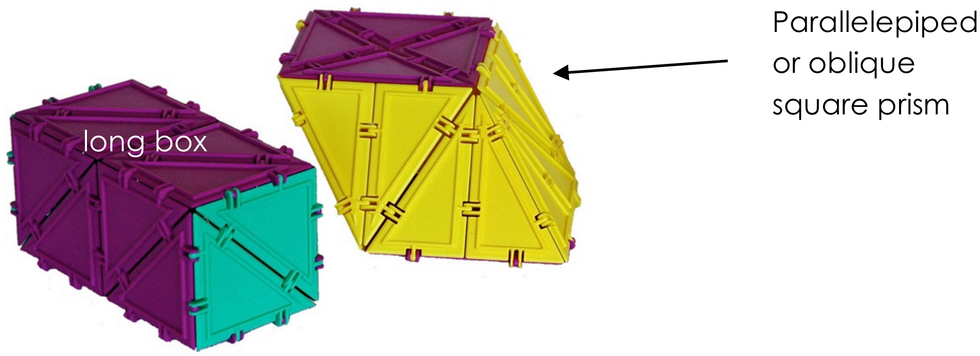


a) Cavalieri's principle
b) Use the result of Exercise 1.

Which object has the greater surface area?

Exercise 3

Prove that the parallelepiped (oblique square prism) has the same volume as the long box in 2 ways.

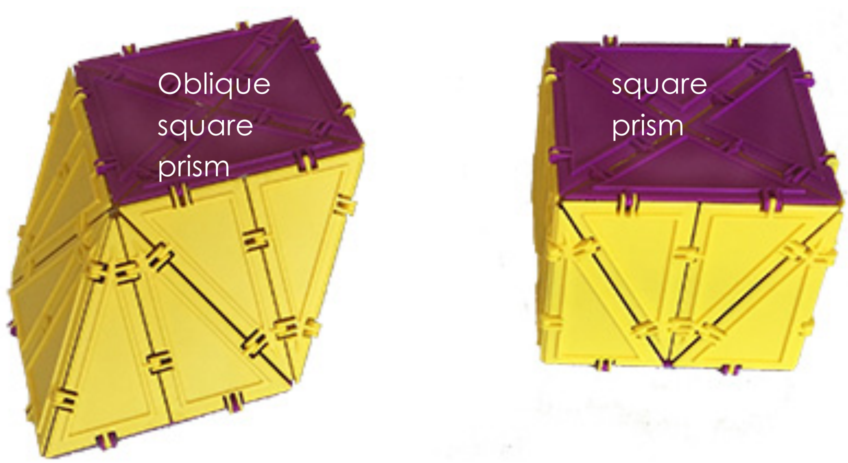


1. Use Cavalieri's principle
2. Use dissection and the results of previous exercises.

Which object has the greater surface area?

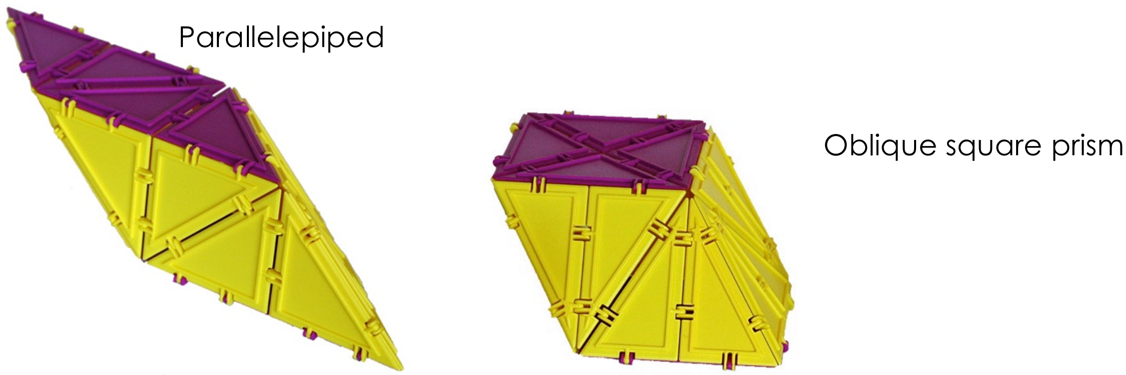
Exercise 4

Prove that the square prism and the oblique square prism have different volumes but the same surface area.



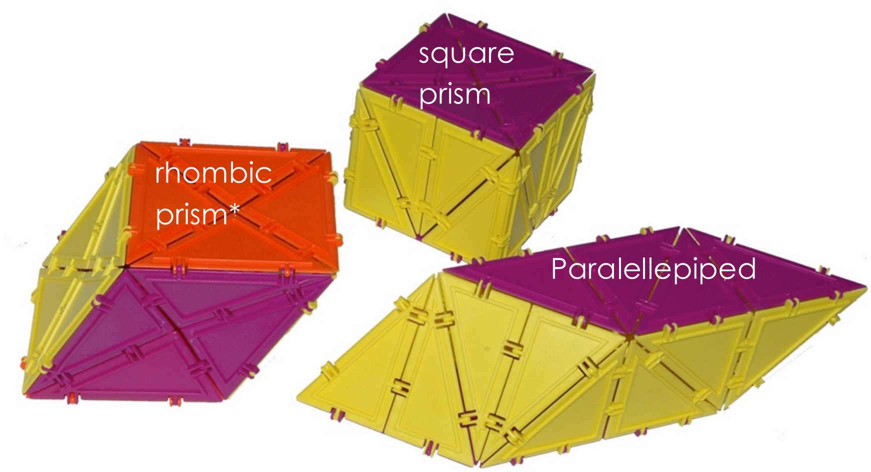
Exercise 5

Prove that the parallelepiped has the same volume as the oblique square prism but a different surface area.



Exercise 6

Determine which two of these objects have the same volume, and which two have the same surface area.



* The rhombic base is the purple rhombus in the picture.

Solutions

Exercise 1

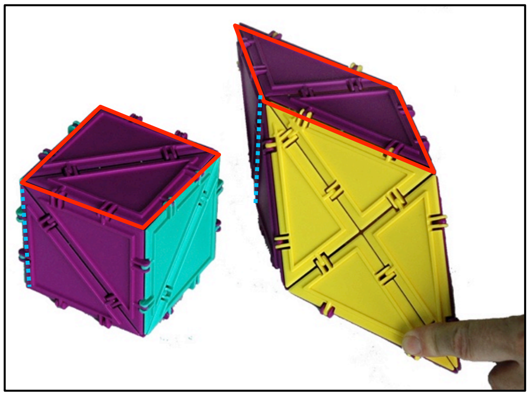
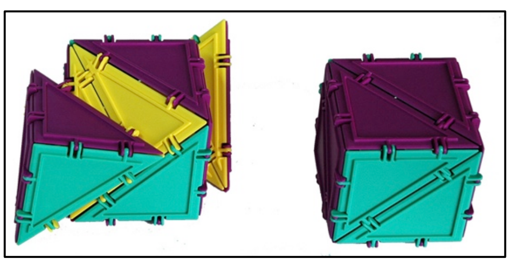


Figure 8

a) From Figure 8, we see that the horizontal cross section of each solid (marked in red) is made of two right isosceles triangles. In one case the triangles form a square, while in the other they form a parallelogram. Of course, the square and the parallelogram have the same area. Also, the solids are the same height, as you can see from the dotted blue lines in Figure 8. In each case, the length of the dotted line, which is the side length of the right isosceles triangle, is the height of the solid. Therefore, by Cavalieri's principle, they have the cube and the parallelepiped have the same volume.



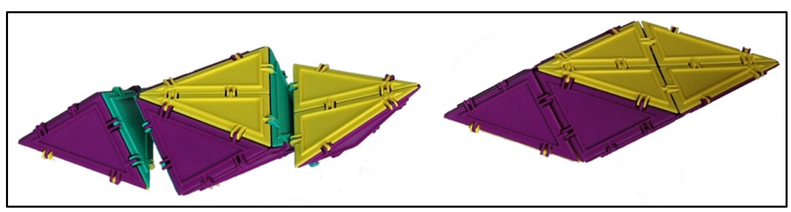


Figure 9

Figure 10

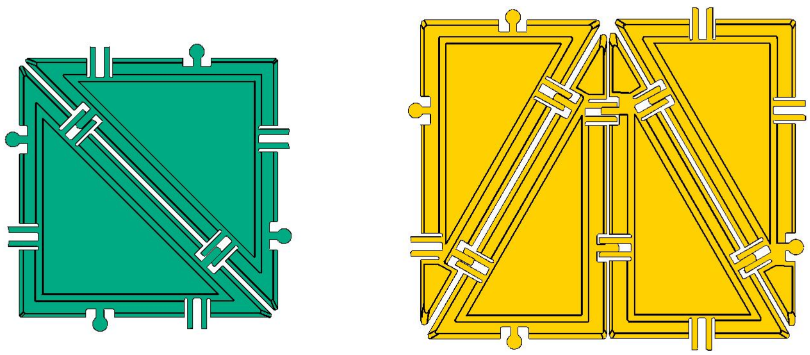
b) You can subdivide the cube into 3 parts as shown in Figure 9 and rearrange them into a parallelepiped as shown in Figure 10.

The cube is made of 12 right isosceles triangles, while the oblique rhombic prism is made of 8 right scalene triangles and 8 right isosceles triangles. "Subtracting" 8 right isosceles triangles from each side we need to compare the area of

4 right isosceles triangles   vs   8 right scalene triangles

We can "divide" both sides by 2 and compare the area of

2 right isosceles triangles   vs   4 right scalene triangles



It is easy to see that the left hand side is a square whose lengths are smaller than the lengths of each of the sides of the rectangle on the right. Therefore, the oblique rhombic prism has greater surface area. We can actually intuit this result by noting that the cube is the more sphere-like of the two objects. Therefore, it has the smaller surface area.

Exercise 2

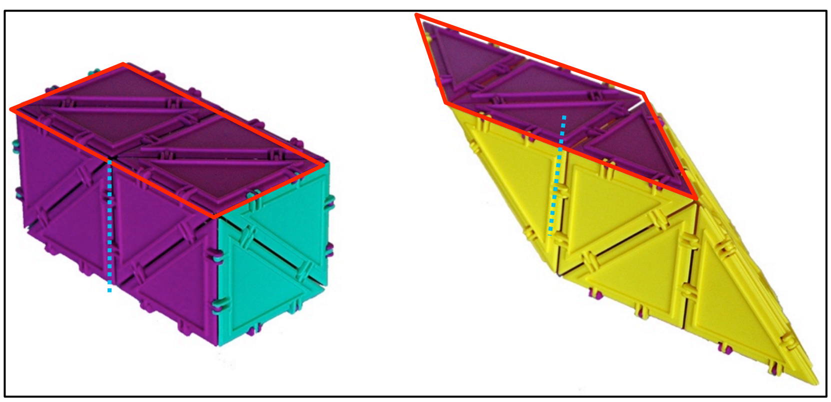


Figure 11

a) From Figure 11 , you can see that the horizontal cross section of each solid (marked in red) is four right isosceles triangles. The height of each solid is equal to the length of one of the legs of the right isosceles triangles (dashed blue line). Therefore, by Cavalieri's principle, they have the same volume.

b) We saw in Exercise 1 that a certain oblique rhombic prism has the same volume as a cube. It so happens that two of these oblique rhombic prisms make the parallelepiped in Figure 11:

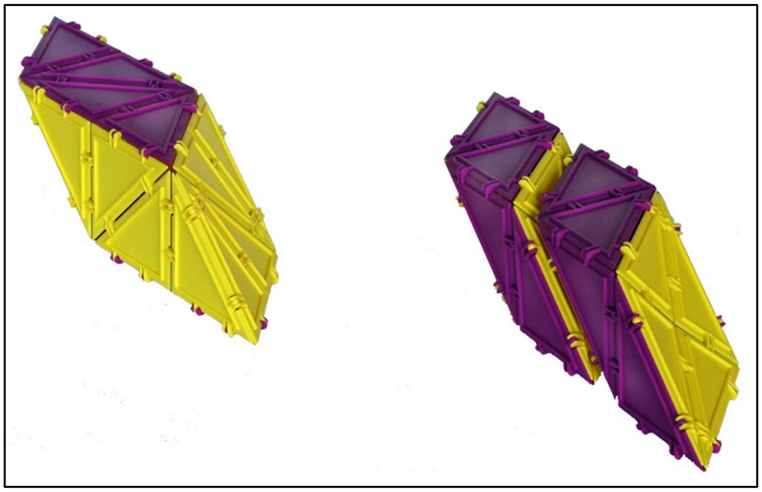


Figure 12

Therefore, two of these oblique rhombic prisms have the same volume as two cubes, which have the same volume as the long box.

The box is made of 20 right isosceles triangles, while the parallelepiped is made of 16 right isosceles triangles and 8 right scalene triangles. If we "subtract" 16 right isosceles triangles from each side, we will be comparing

4 right isosceles triangles vs 8 right scalene triangles

We have already seen in a previous example that the 8 right scalene triangles have the greater area. Therefore, the parallelepiped has the greater surface area.

Exercise 3

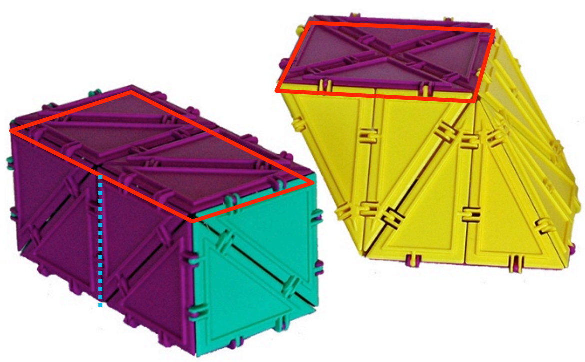


Figure 13

a) From Figure 13 you can see that the horizontal cross section of each solid (marked in red) is four right isosceles triangles. The height of each solid is equal to the length of one of the legs of the right isosceles triangles (dashed blue line). This is a bit more challenging to see with the solid on the right (but easier to see in Figure 14, right). By Cavalieri's principle, the long box and oblique square prism have the same volume.

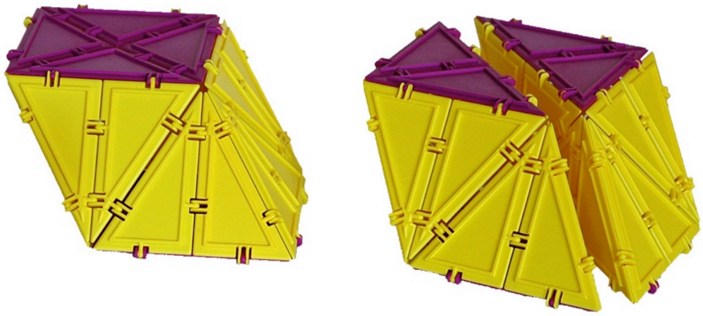


Figure 14

b) As seen in Figure 14, we can split the oblique square prism down the middle. Then we can reassemble these halves into the parallelepiped of Figure 11:

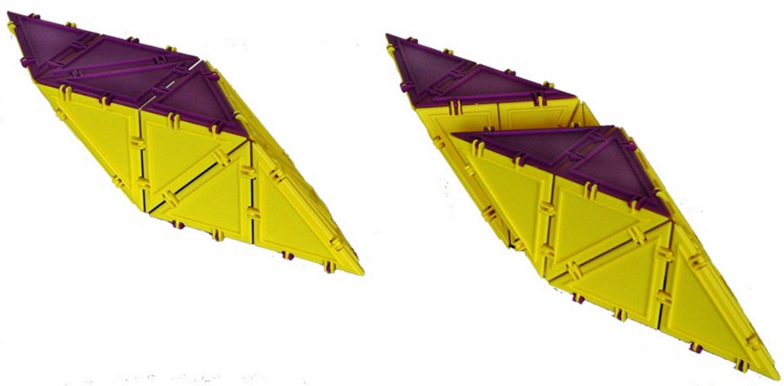
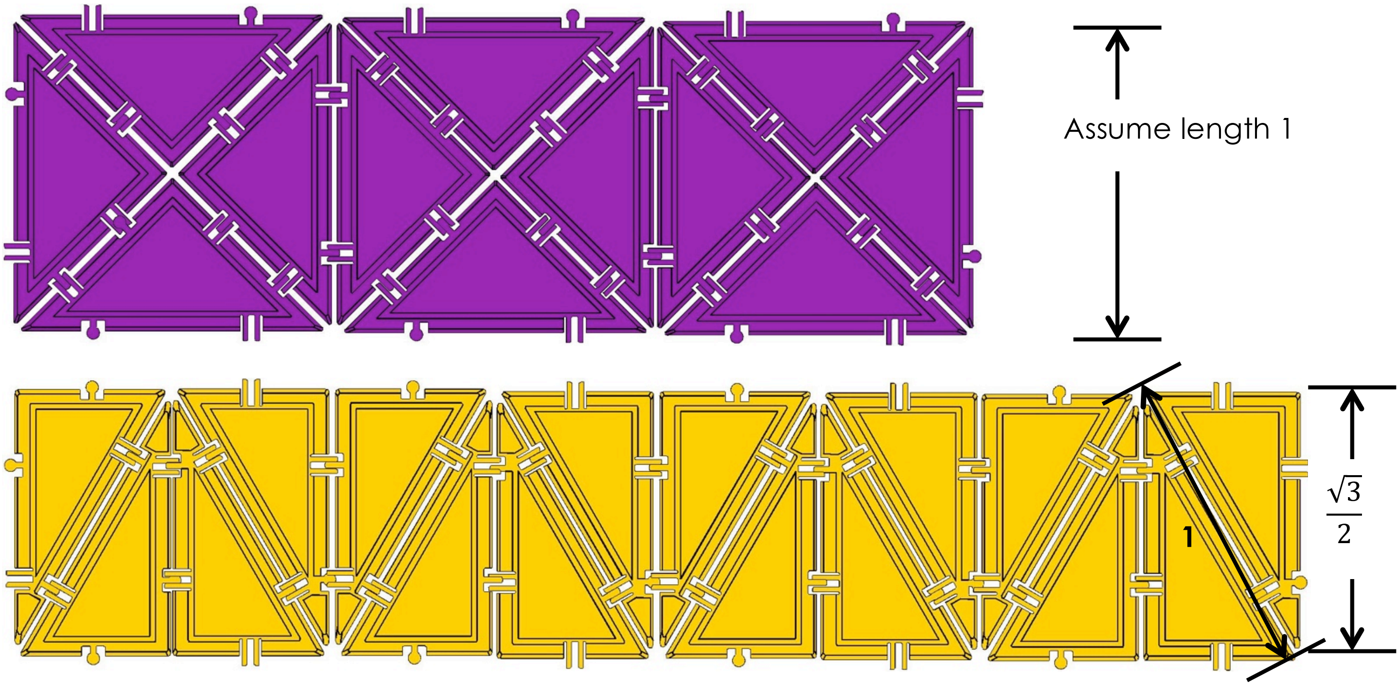


Figure 15

Now, we have already proved in Exercise 2 that the parallelepiped of Figure 15, left, has the same volume as the long box. Since the square oblique prism has the same volume as the parallelepiped of Figure 15, it also has the same volume as the box.

The long box has surface area equal to 20 right isosceles triangles, while the oblique square prism has surface area equal to 8 right isosceles triangles and 16 right scalene triangles. We "subtract" 8 right isosceles triangles from each side to end up comparing the areas of

12 right isosceles triangles vs 16 right scalene triangles



Let us compute the areas of each of these quantities assuming that the hypotenuse of each type of triangle has length 1.

The area of 12 right isosceles triangles is 3, while the area of the 16 right scalene triangles is $4\times\frac{\sqrt{3}}{2} = 2\sqrt{3} > 3$. So the oblique square prism has a greater surface area.

Exercise 4

While the horizontal cross sections of each solid ar the same, their heights are different.

Each solid is made of 8 right isosceles triangles and 16 right scalene triangles, so they have the same surface area.

Exercise 5

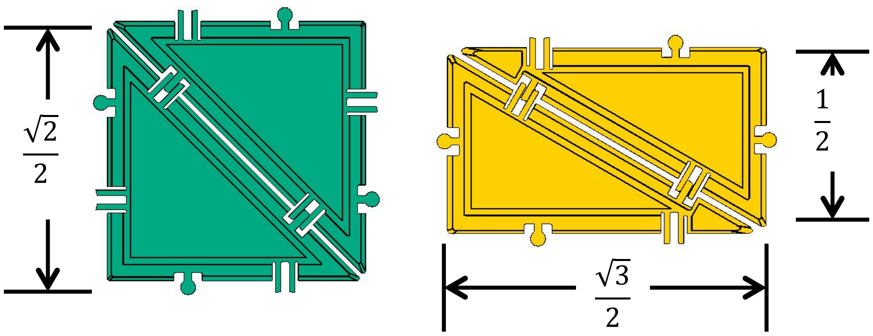
It was shown in Exercise 3 and Exercise 4 that each of these solids has the same volume as the long box. Therefore, they have the same volume.

The parallelepiped is made of 16 right isosceles triangles and 8 right scalene triangles, while the oblique square prism is made of 8 right isosceles triangles and 16 right scalene triangles.

To compare these quantities, let us "subtract" 8 right isosceles triangles and 8 right scalene triangles from each side. Then we are comparing the areas of

8 right isosceles triangles vs 8 right scalene triangles

So we're down to finding out which has larger area: a right isosceles triangle or a right scalene triangle, when both have a hypotenuse of the same length. It is easier to compare them in pairs, since two of each triangles makes a square or a rectangle.



As before, let's assume the hypotenuse has unit length. Connecting two right isosceles together, we get a square of area $\frac{1}{2}$. On the other hand, connecting two right scalene triangles together gives us a rectangle of area $\frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} < \frac{1}{2}$. Interestingly, the areas of the two triangles are very close to one another. $\frac{\sqrt{3}}{4} \approx 0.43$, which is quite close to $\frac{1}{2} = 0.5$.

Exercise 6

All three solids have a cross sectional area equal to the area of 4 right isosceles triangles as shown in the picture. The height of the rhombic prism and the square prism are both $\frac{\sqrt{3}}{2}$, assuming the isosceles and scalene triangles each have a hypotenuse of unit length. So the rhombic prism and the square prism have the same volume by Cavalieri's principle. However, the parallelepiped has height equal to $\frac{\sqrt{2}}{2}$, so it has a different volume.

The parallelepiped and the rhombic prism are both made of 16 right isosceles triangles and 8 right scalene triangles, so they have the same surface area. The square prism, on the other hand, is made of 8 right isosceles triangles and 16 right scalene triangles, so it has a different surface area from the other two figures.

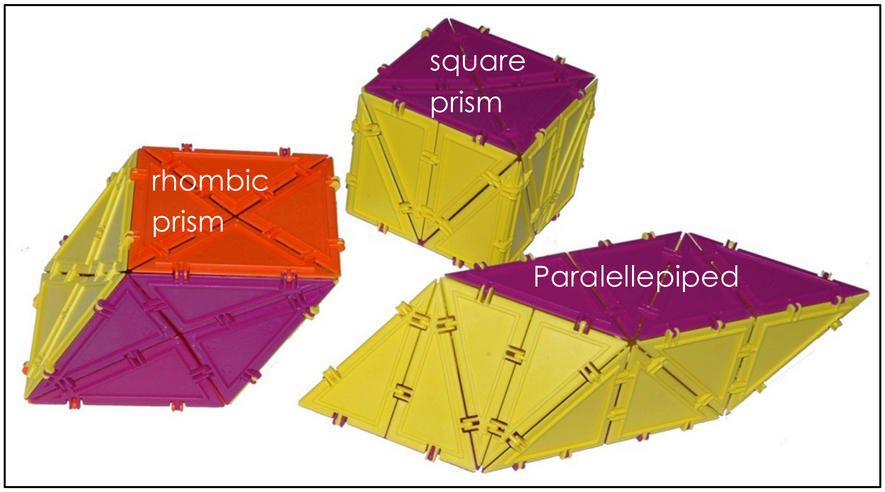


Figure 16

